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Added another proof in Section 2 and also a new Appendix

Integration with Functions of a Quaternionic Variable

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Abstract

Recent innovations in the differential calculus for functions of non-commuting variables, beginning with a quaternionic variable, are now extended to consider some integration.

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1 The Differential

In a recent paper [1] I showed how to expand

$$F(x+\delta) = F(x) + \mathcal{D}F(x) + O(\delta^2) \tag{1.1}$$

when both x and δ were general quaternionic variables, and thus did not commute with each other; and we found the general formula,

$$\mathcal{D}F(x) = F'(x)\delta_\parallel + [F(x) - F(x^*)](x - x^*)^{-1}\,\delta_\perp. \tag{1.2}$$

The specifics of how to construct the two components of $\delta = \delta_\parallel + \delta_\perp$ are,

$$\delta_\parallel = \frac{1}{2}(\delta - u_x\,\delta\,u_x), \qquad \delta_\perp = \frac{1}{2}(\delta + u_x\,\delta\,u_x), \tag{1.3}$$

where u_x is a unit imaginary that depends upon the location of $x = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3$ as a point in a four-dimensional Euclidean space:

$$u_x = (i\xi_1 + j\xi_2 + k\xi_3)/r, \qquad r = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}. \tag{1.4}$$

In another recent paper [2] similar results were obtained for a different kind of non-commuting variables - matrices over the complex numbers.

This may be called the beginning of non-commutative calculus - the differential part. Now we want to try looking at the integral part of that calculus.

2 The Integral

In ordinary calculus of functions of the real variable t, we know what is meant by an integral, such as $\int f(t)\,dt$. But when we consider quaternionic (or other non-commuting) variables it is unclear even how to write such an expression. (See further, Appendix A.)

Alternatively, we can start with the defining relation between the integral and the differential:

$$\int_a^b df(t) = \int_a^b \frac{df(t)}{dt}\,dt = f(b) - f(a); \tag{2.1}$$

and this is what we shall generalize for our non-commuting quaternionic variable x as,

$$\int_a^b \mathcal{D}F(x) = F(x_b) - F(x_a). \tag{2.2}$$

We define this integral as an additive operation along a path in that four-dimensional space of the real variables ξ,

$$x = x_{path}(s), \qquad x_{path}(0) = x_a, \qquad x_{path}(1) = x_b \tag{2.3}$$

where s is a real continuous parameter.

Next, we subdivide that path, whatever it may be, into a large number of infinitesimal increments.

$$\int_a^b = \sum_{n=1}^{n=N} \int^{(n)}, \qquad \int^{(n)} = \int_{x_{n-1}}^{x_n}, \qquad n = 1, \ldots, N \tag{2.4}$$

where $x_0 = x_a$ and $x_N = x_b$.

In any segment of this path we choose the line of integration, with the integrand $\mathcal{D}F(x)$, to be the sum of two infinitesimal parts:

$$x_n - x_{n-1} = \delta = \delta_{\|} + \delta_{\perp}. \tag{2.5}$$

The first part is "parallel" to the direction of x at that point, giving the contribution

$$\int_{\|} \mathcal{D}F(x) = F'(x)\delta_{\|}. \tag{2.6}$$

Then the second part is "perpendicular", giving the contribution

$$\int_{\perp} \mathcal{D}F(x) = [F(x) - F(x^*)](x - x^*)^{-1}\,\delta_{\perp}. \tag{2.7}$$

The sum of these two parts is thus nothing other than

$$F(x_n) - F(x_{n-1}) \tag{2.8}$$

to first order in the interval δ. The entire sum then results in Eq. (2.2).

As an alternative to this "staircase" construction of the integration, we can look at the following model. Say that the path of integration is a straight line

$$x = \alpha + i\beta + j\gamma s \tag{2.9}$$

where s is a real variable that goes from 0 to 1. Then we use the definitions Eqs. (1.3), (1.4) to calculate the quantities in $\mathcal{D}F(x)$. This is a rather tedious procedure; but I have carried it out for $F(x) = x, x^2, x^3$ and found that the formula Eq. (2.2) is verified.

Another general proof can proceed as follows. If we start with the coordinate along the path $x(s) = \xi_0(s) + i\xi_1(s) + j\xi_2(s) + k\xi_3(s)$, then we can simply write,

$$\mathcal{D}x(s) = ds\frac{dx(s)}{ds} \tag{2.10}$$

since there is no commutativity problem in this representation. It is also true that we can express any function as

$$F(x(s)) = A(s) + B(s)\, x(s) \tag{2.11}$$

where A and B are real functions, the only quaternions being in the single factor $x(s)$. We then see that the integral becomes quite ordinary:

$$\int_a^b \mathcal{D}F(x(s)) = \int_0^1 ds\frac{dF(x(s))}{ds} = F(x(s))|_0^1 = F(x_b) - F(x_a). \tag{2.12}$$

In [1] it was shown that this differential operator $\mathcal{D}$ obeys the Leibnitz rule; and thus we get the identity, usually called "integration by parts",

$$\int_a^b F(x)\, \mathcal{D}G(x) = F(x_b)G(x_b) - F(x_a)G(x_a) - \int_a^b (\mathcal{D}F(x))\, G(x). \tag{2.13}$$

Loosely speaking, integration is the inverse of differentiation. What we see in Eqs. (2.1) and (2.2) is one statement of that relationship. But there is also the alternate form, which is stated for real variables as

$$\frac{d}{dt} \int^t f(t')\, dt' = f(t). \tag{2.14}$$

For our quaternionic variables we start by looking at

$$\mathcal{D}_x \int^x \mathcal{D}_{x'}\, F(x') \tag{2.15}$$

and then apply the first differential operator to the coordinate x in two parts: first the $\delta_{\parallel}$ part and then the $\delta_{\perp}$ part. The result is just the integrand evaluated at the point x:

$$= \mathcal{D}_x\, F(x); \tag{2.16}$$

and this is just what we should expect from the right hand side of Eq. (2.2), with x_b replaced by x.

3 Discussion

Following what was stated in the earlier work, [1], we do require the functions $F(x)$ to be real analytic functions along the path of integration.

Our main result Eq. (2.2) implies that the result of the integration depends only on the end points and is independent of the path. This is true if we also require that the function $F(x)$ be single valued. Then, we have the result that the integral over any closed path, ending up at the same point where it started, is zero. For more discussion of this, see Appendix B.

Appendix A

If we look at the real integral and try to guess how to generalize it to the non-commutative quaternions, we might start with,

$$\int f(t)\,dt \xrightarrow{?} \frac{1}{2}\int (dx\,F(x) + F(x)\,dx)\,; \tag{A.1}$$

but why should dx only appear on the outside; why not also in the middle of the function $F(x)$?

Let's try a most symmetrical arrangement with the function $F(x) = x^n$:

$$\int t^n\,dt \xrightarrow{?} \frac{1}{n+1}\int (dx\,x^n + x\,dx\,x^{n-1} + x^2\,dx\,x^{n-2} + \ldots + x^n\,dx). \tag{A.2}$$

But we can recognize that the long expression in parentheses on the right hand side of this is nothing other than $\mathcal{D}x^{n+1}$:

$$\mathcal{D}F(x) \equiv F(x+dx) - F(x), \;\; to\ first\ order\ in\ dx. \tag{A.3}$$

So we would then write,

$$\int t^n\,dt \longrightarrow \frac{1}{n+1}\int \mathcal{D}x^{n+1} = \frac{x^{n+1}}{n+1}, \tag{A.4}$$

using our defining Eq. (2.2). Now, this looks quite familiar.

We can extend this to any power series and thus offer the following rule. For any analytic function of a real variable $f(t)$, for which we know the integral,

$$\int f(t)\,dt = h(t), \tag{A.5}$$

we can make the correspondence to quaternionic integration as follows:

$$\int f(t)\, dt \longrightarrow \int \mathcal{D}h(x) = h(x). \tag{A.6}$$

While this may look trivial for real and complex variables, it is something new for non-commuting variables. This is because we have carefully defined and studied the operator $\mathcal{D}$.

Appendix B

In the familiar study of functions of a complex variable, we have the rule about integrals around a pole,

$$\oint \frac{g(z)}{z}\, dz = 2\pi i\, g(0) \tag{B.1}$$

for well behaved functions g. How does this square with our general statement above that any integral over a closed path would be zero?

Our formulation above, as applied to a complex variable, reads

$$\int_a^b \frac{df(z)}{dz}\, dz = f(b) - f(a) \tag{B.2}$$

and this should be zero when $f(z)$ is analytic and single valued over the contour of integration and that contour is closed: $a = b$. The way to get the integral (B.2) to look like the integral (B.1) is to choose the function $f(z) = ln\ z$. But this function is not single valued. In going once around the origin, $ln\ z$ changes by exactly the amount $2\pi i$. So we have agreement between the results of integrals (B.1) and (B.2).

References

[1] C. Schwartz, arXiv:0803.3782 [math.FA]

[2] C. Schwartz, arXiv:0804.2869 [math.FA]